

Constrained BSDE and Viscosity Solutions of Variation Inequalities*

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Abstract. In this paper, we study the relation between the smallest g -supersolution of constraint backward stochastic differential equation and viscosity solution of constraint semilinear parabolic PDE, i.e. variation inequalities. And we get an existence result of variation inequalities via constraint BSDE, and prove a uniqueness result under certain condition.

Keywords: Backward stochastic differential equation with a constraint, viscosity solution, variation inequality.

1 Introduction

El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) studied the problem of BSDE (backward stochastic differential equation) with reflection, which is, a standard BSDE with an additional continuous, increasing process in this equation to keep the solution above a certain given continuous boundary process. This increasing process must be chosen in certain minimal way, i.e. an integral condition, called Skorohod reflecting condition (cf. [?]), is satisfied. It was proved in this paper that the solution of the reflected BSDE associated to a terminal condition ξ , a coefficient g and a lower reflecting obstacle L , is the smallest supersolution of BSDE with same parameter (ξ, g) , which dominates the given boundary process L . Then in same paper, they give a probabilistic interpretation of viscosity solution of variation inequality by the solution of reflected BSDEs.

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An important application of the constrained BSDE is the pricing of contingent claims with constraint of portfolios, i.e. portfolios of an asset is constrained in a given subset. In this case the solution (y, z) of the corresponding reflected BSDE must remain in this subset. This problem was studied by Karatzas and Kou (cf. [?]), then by [4] and [2].

The most general case of the constraint Φ , which is described by a Lipschitz continuous function, is first studied in [6]. Author proved that under the Lipschitz condition of coefficient g , the smallest supersolution of BSDE with coefficient g and constraint $\Phi \geq 0$ exists, if there exists a special solution of this constraint problem.

The main conditions of our paper is same as [6]: g is a Lipschitz function and the constraint $\Phi(\omega, t, x, y, z)$, $t \in [0, T]$ is a Lipschitz continuous function. In this paper we study the relation between the smallest g -supersolution and viscosity solution of constraint semilinear parabolic PDE, i.e. variation inequalities. And we get an existence result of variation inequalities via constraint BSDE, and prove a uniqueness result under certain condition.

2 Preliminaries and Constraint BSDEs

Let (Ω, \mathcal{F}, P) be a probability space, and $B = (B_1, B_2, \dots, B_d)^T$ be a d -dimensional Brownian motion defined on $[0, \infty)$. We denote by $\{\mathcal{F}_t; 0 \leq t < \infty\}$ the natural filtration generated by this Brownian motion B :

$$\mathcal{F}_t = \sigma\{\{B_s; 0 \leq s \leq t\} \cup \mathcal{N}\},$$

where \mathcal{N} is the collection of all P -null sets of \mathcal{F} . The Euclidean norm of an element $x \in \mathbb{R}^m$ is denoted by $|x|$. We also need the following notations for $p \in [1, \infty)$:

- $\mathbf{L}^p(\mathcal{F}_t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued } \mathcal{F}_t\text{-measurable random variables } X \text{ s.t. } E[|X|^p] < \infty\};$
- $\mathbf{L}_{\mathcal{F}}^p(0, t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued and } \mathcal{F}_t\text{-progressively measurable processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E \int_0^t |\varphi_s|^p ds < \infty\};$
- $\mathbf{D}_{\mathcal{F}}^p(0, t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued and RCLL } \mathcal{F}_t\text{-progressively measurable processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E[\sup_{0 \leq s \leq t} |\varphi_s|^p] < \infty\};$
- $\mathbf{A}_{\mathcal{F}}^p(0, t) := \{\text{increasing processes } A \text{ in } \mathbf{D}_{\mathcal{F}}^p(0, t; \mathbb{R}) \text{ with } A(0) = 0\}.$

When $m = 1$, they are denoted by $\mathbf{L}^p(\mathcal{F}_t)$, $\mathbf{L}_{\mathcal{F}}^p(0, t)$ and $\mathbf{D}_{\mathcal{F}}^p(0, t)$, respectively. We are mainly interested in the case $p = 2$. In this paper, we consider BSDE on the interval $[0, T]$, with a fixed $T > 0$.

We put the BSDE with a constraint into Markovian framework. Consider the following forward SDE,

$$\begin{aligned} dX_s^{t,x} &= b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \quad t \leq s \leq T, \\ X_t^{t,x} &= x. \end{aligned} \tag{1}$$

where $b : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^d$, $\sigma : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^{d \times d}$ are continuous mappings, satisfying

- (i) b and σ are continuous in t
- (ii) $|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq k(|x - x'|)$, $dP \times dt$ a.s.

for some $k > 0$, and for all $x, x' \in \mathbf{R}^d$. And for each $(t, x) \in [0, T] \times \mathbf{R}^d$, $\{X_s^{t,x}; t \leq s \leq T\}$ is denoted as the unique solution of SDE (1).

Let g be a coefficient $g(t, x, y, z) : [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R}$, which satisfies the following assumptions: there exists a constant $\mu > 0$, $p \in \mathbf{N}$ such that, for each x in \mathbf{R}^d , y, y' in \mathbf{R} and z, z' in \mathbf{R}^d , we have

- (i) $|g(t, x, 0, 0)| \leq \mu(1 + |x|^p)$
- (ii) $|g(t, x, y, z) - g(t, x, y', z')| \leq \mu(|y - y'| + |z - z'|)$, $dP \times dt$ a.s. (2)

Our BSDE with a constraint is

$$\begin{aligned} -dY_s^{t,x} &= g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + dA_s^{t,x} - Z_s^{t,x}dW_s, \\ Y_T^{t,x} &= \Psi(X_T^{t,x}), \quad \text{with } \Phi(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \geq 0, d\mathbf{P} \times dt \text{-a.s..} \end{aligned} \quad (3)$$

Here $\Psi : \mathbf{R}^d \rightarrow \mathbf{R}$, has at most polynomial growth at infinity. $\Phi : [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R}$, which plays a role of constraint in this paper, satisfying: there exists a constant $\mu_2 > 0$, such that, for each $x \in \mathbf{R}^d$, $y, y' \in \mathbf{R}$ and $z, z' \in \mathbf{R}^d$, we have

- (i) $|\Phi(t, x, 0, 0)| \leq \mu_2(1 + |x|^p)$
- (ii) $|\Phi(t, x, y, z) - \Phi(t, x, y', z')| \leq \mu_2(|y - y'| + |z - z'|)$, $dP \times dt$ a.s. (4)
- (iii) $y \rightarrow \Phi(t, x, \cdot, z)$ and $z \rightarrow \Phi(t, x, y, \cdot)$ are continuous.

The constraint Φ is an equivalent form of the constraint we have discussed before, as [3], [6] and [9].

Definition 2.1. The solution of (3) is $(Y_s^{t,x}, Z_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$ defined as the smallest g -supersolution constrained by $\Phi \geq 0$, i.e. $Y^{t,x} \in \mathbf{D}_{\mathcal{F}}^2(t, T)$ and there exist a predictable process $Z^{t,x} \in \mathbf{L}_{\mathcal{F}}^2(t, T; \mathbf{R}^d)$ and an increasing RCLL process $A^{t,x} \in \mathbf{A}_{\mathcal{F}}^2(t, T)$ such that (3) is satisfied and if there is another process $Y^{t,x'} \in \mathbf{D}_{\mathcal{F}}^2(t, T)$, with $(Z^{t,x'}, A^{t,x'}) \in \mathbf{L}_{\mathcal{F}}^2(t, T; \mathbf{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(t, T)$, satisfying (3), then we have $Y_s^{t,x'} \geq Y_s^{t,x}$.

The following theorem of the existence of the smallest solution was obtained in [6].

Theorem 2.1. Suppose that $\xi \in \mathbf{L}^2(\mathcal{F}_T)$, the function g satisfies (2) and the constraint Φ satisfies (4). We assume that (**H**) there is one g -supersolution $y' \in \mathbf{D}_{\mathcal{F}}^2(0, T)$, constrained by $\Phi \geq 0$:

$$\begin{aligned} y'_t &= \xi + \int_t^T g(s, y'_s, z'_s)ds + A'_T - A'_t - \int_t^T z'_s dB_s, \\ A' &\in \mathbf{A}_{\mathcal{F}}^2(0, T), \quad \Phi(t, y'_t, z'_t) \geq 0, \quad dP \times dt \text{ a.s.} \end{aligned} \quad (5)$$

Then there exists the smallest g -supersolution $y \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ constrained by $\Phi \geq 0$, with the terminal condition $y_T = \xi$, i.e. there exists a triple $(y_t, z_t, A_t) \in \mathbf{D}_{\mathcal{F}}^2(t, T) \times \mathbf{L}_{\mathcal{F}}^2(t, T; \mathbf{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(t, T)$, such that

$$\begin{aligned} y_t &= \xi + \int_t^T g(s, y_s, z_s)ds + A_T - A_t - \int_t^T z_s dB_s, \\ A &\in \mathbf{A}_{\mathcal{F}}^2(0, T), \quad \Phi(t, y_t, z_t) \geq 0, \quad dP \times dt \text{ a.s.} \end{aligned}$$

Moreover, this smallest g -supersolution is the limit of a sequence of g^n -solutions with $g^n = g + n\Phi^-$, where the convergence is in the following sense:

$$y_t^n \nearrow y_t, \text{ with } \lim_{n \rightarrow \infty} E[|y_t^n - y_t|^2] = 0, \quad \lim_{n \rightarrow \infty} E \int_0^T |z_t - z_t^n|^p dt = 0, p \in [1, 2), \quad (6)$$

$$A_t^n : = \int_0^t (g + n\Phi^-)(s, y_s^n, z_s^n) ds \rightarrow A_t \text{ weakly in } \mathbf{L}^2(\mathcal{F}_t), \quad (7)$$

where z and A are the corresponding martingale part and increasing part of y , respectively.

And we recall an interesting proposition proved in [9].

Proposition 2.1. *A process $y \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ is the smallest g -supersolution on $[0, T]$ constraint by Φ with $y_T = \xi$, if and only if for all $m \geq 0$, it is a $(g + m\Phi)$ -supersolution on $[0, T]$ with $y_T = \xi$.*

3 Relation between BSDE with a constraint and PDE

In the following, we assume that **(H)** holds, and denote the smallest solution of (3) by $(Y_s^{t,x}, Z_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$. Define

$$u(t, x) := Y_t^{t,x}.$$

The variation inequality we concerned is

$$\min\{-\partial_t u - F_0(t, x, u, Du, D^2u), \Phi(x, u, \sigma^T(x)Du)\} = 0,$$

where $F_0(t, x, u, q, S) := \frac{1}{2} \sum_{i,j=1}^n [\sigma\sigma^T]_{ij}(t, x)S_{ij} + \langle b(t, x), q \rangle + g(t, x, u, \sigma^T(t, x)q)$. We study this problem by the following penalization approach: for each $\alpha \geq 0$,

$$\begin{aligned} -dY_s^{t,x,\alpha} &= (g + \alpha\Phi^-)(s, X_s^{t,x}, Y_s^{t,x,\alpha}, Z_s^{t,x,\alpha}) ds - Z_s^{t,x,\alpha} dW_s, \\ Y_T^{t,x,\alpha} &= \Psi(X_T^{t,x}). \end{aligned}$$

Define

$$u_{\alpha}(t, x) := Y_t^{t,x,\alpha}. \quad (8)$$

Then by theorem 2.1, we have

$$u_{\alpha}(t, x) \nearrow u(t, x), \quad (t, x) \in [0, T] \times \mathbf{R}^n, \text{ as } \alpha \rightarrow \infty. \quad (9)$$

We introduce the following penalized PDE

$$\partial_t u + F_{\alpha}(t, x, u, Du, D^2u) = 0, \quad \forall \alpha = 1, 2, \dots, \quad (10)$$

where $F_{\alpha}(t, x, u, q, S) := \frac{1}{2} \sum_{i,j=1}^n [\sigma\sigma^T]_{ij}(t, x)S_{ij} + \langle b(t, x), q \rangle + (\alpha\Phi^- + g)(t, x, u, \sigma^T(t, x)q)$.

To introduce the definition of viscosity solution. First we need the notions of parabolic superjet and subjet.

Definition 3.1. For a function $u \in LSC([0, T] \times R^n)$ (resp. $USC([0, T] \times R^n)$), we define the parabolic superjet (resp. parabolic subjet) of u at (t, x) by $\mathcal{P}^{2,+}u(t, x)$ (resp. $\mathcal{P}^{2,-}u(t, x)$), the set of triples $(p, q, X) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^n$, satisfying

$$\begin{aligned} u(s, y) &\leq \text{(resp. } \geq \text{)} u(t, x) + p(s - t) + \langle q, y - s \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle \\ &\quad + o(|s - t| + |y - x|^2). \end{aligned}$$

Then we have

Definition 3.2. A function $u \in LSC([0, T] \times R^n)$ (resp. $USC([0, T] \times R^n)$) is called a viscosity supersolution (resp. subsolution) of $\partial_t u + F_\alpha = 0$ if for each $(t, x) \in (0, T) \times R^n$, for any $(p, q, X) \in \mathcal{P}^{2,-}u(t, x)$ (resp. $(p, q, X) \in \mathcal{P}^{2,+}u(t, x)$), we have

$$p + F_\alpha(t, x, u(t, x), q, X) \leq 0, \text{(resp. } \geq 0).$$

The following result can be found in [5].

Proposition 3.1. We assume (2) and Ψ has at most polynomial growth at infinity. Then for each $\alpha = 1, 2, \dots$, the function $u_\alpha \in C([0, T] \times R^n)$ defined by (8) is the viscosity solution of $\partial_t u_\alpha + F_\alpha = 0$.

Now we return to the variation inequality

$$\min\{-\partial_t u - F_0(t, x, u, Du, D^2u), \Phi(x, u, \sigma^T(x)Du)\} = 0. \quad (11)$$

The solution of this equation may be not continuous, so we need the definition of discontinuous viscosity solution. For a given locally bounded function v , we define its upper and lower semicontinuous envelope of v , denoted as v^* and v_* respectively, where

$$v^*(t, x) = \limsup_{t' \rightarrow t, x' \rightarrow x} v(t', x'), v_*(t, x) = \liminf_{t' \rightarrow t, x' \rightarrow x} v(t', x').$$

Then

Definition 3.3. (i) A locally bounded function u is called a viscosity supersolution (11) if for each $(t, x) \in (0, T) \times \mathbf{R}^n$, for any $(p, q, X) \in \mathcal{P}^{2,-}u_*(t, x)$, then we have

$$\min\{-p - F_0(t, x, u_*, q, X), \Phi(x, u_*, \sigma^T(x)q)\} \geq 0, \quad (12)$$

i.e. we have both $\Phi(x, u_*, \sigma^T(x)q) \geq 0$ and

$$-p - F_0(t, x, u_*, q, X) \geq 0.$$

(ii) A locally bounded function u is called a viscosity subsolution of (11), if for each $(t, x) \in (0, T) \times \mathbf{R}^n$, for any $(p, q, X) \in \mathcal{P}^{2,+}u^*(t, x)$, then we have

$$\min\{-p - F_0(t, x, u^*, q, X), \Phi(x, u^*, \sigma^T(x)q)\} \leq 0, \quad (13)$$

i.e. for $(t, x) \in (0, T) \times R^n$ where $\Phi(x, u^*, \sigma^T(x)q) > 0$, we have

$$-p - F_0(t, x, u^*, q, X) \leq 0.$$

(iii) A locally bounded function u is called a viscosity solution of (11), if it is both viscosity super- and subsolution.

We recall the function $u(t, x)$ is denoted by $u(t, x) := Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$ is the smallest solution of BSDE (3) constraint by $\Phi \geq 0$. And such solution exists. Our first result is following.

Proposition 3.2. *For each $\alpha = 1, 2, \dots$, $u(t, x)$ is a discontinuous viscosity supersolution of $\partial_t u + F_\alpha = 0$.*

Proof. It is an application of Proposition 2.1 and the fact that a g -supersolution relate to viscosity supersolution. \square

Then we have

Theorem 3.1. *The function u is a discontinuous viscosity solution of (11).*

Proof. From the above discussion, we know that for each $\alpha = 1, 2, \dots$, u_α , defined by $u_\alpha(t, x) := Y_t^{t,x,\alpha}$, is a viscosity solution of $\partial_t u_\alpha + F_\alpha = 0$. And $u_\alpha(t, x) \nearrow u(t, x)$, so $u(t, x)$ is lower semicontinuous, i.e. $u(t, x) = u_*(t, x)$.

We now prove that u is a subsolution of (11). Let (t, x) be a point such that $\Phi(t, x, u^*(t, x), \sigma^T q) > 0$, and $(p, q, X) \in \mathcal{P}^{2,+} u^*(t, x)$.

By Lemma 6.1 in [1], there exist sequences

$$\alpha_j \rightarrow \infty, (t_j, x_j) \rightarrow (t, x), (p_j, q_j, X_j) \in \mathcal{P}^{2,+} u_{\alpha_j}(t, x),$$

such that

$$(u_{\alpha_j}(t_j, x_j), p_j, q_j, X_j) \rightarrow (u^*(t, x), p, q, X).$$

While for each j ,

$$\begin{aligned} & -p_j - F_\alpha(t_j, x_j, u_{n_j}(t_j, x_j), q_j, X_j) \\ &= -p_j - [F_0(t_j, x_j, u_{n_j}(t_j, x_j), q_j, X_j) + \alpha \Phi^-(t_j, x_j, u_{\alpha_j}(t_j, x_j), \sigma^T q_j)] \leq 0. \end{aligned}$$

From the assumption that $\Phi(t, x, u^*(t, x), \sigma^T q) > 0$, continuity assumption of Φ and convergence of u_α , it follows for j large enough, $\Phi^-(t_j, x_j, u_{\alpha_j}(t_j, x_j), \sigma^T q_j) > 0$. Hence taking limit in the above inequality, we get

$$-p - F_0(t, x, u^*(t, x), q, X) \leq 0,$$

we prove that u is viscosity subsolution of (11).

Then we conclude by proving that u is a viscosity supersolution of (11). Let $(t, x) \in [0, T] \times \mathbf{R}^n$, and $(p, q, X) \in \mathcal{P}^{2,-} u_*(t, x)$, by proposition 3.2, we know that u is a discontinuous viscosity supersolution of $\partial_t u + F_\alpha = 0$, for each $\alpha \geq 0$, i.e.

$$\begin{aligned} & -p - F_\alpha(t, x, u_*(t, x), q, X) \\ &= -p - F_0(t, x, u_*(t, x), q, X) - \alpha \Phi^-(t, x, u_*, \sigma^T(t, x)q) \geq 0. \end{aligned}$$

By the arbitrary of α , we have

$$-p - F_0(t, x, u_*(t, x), q, X) \geq 0 \text{ and } \Phi^-(t, x, u_*, \sigma^T(t, x)q) = 0,$$

i.e. $\Phi(t, x, u_*(t, x), \sigma^T q) \geq 0$. \square

Then we consider the uniqueness of the solution. First, we have a characterization property of $u(t, x)$.

Proposition 3.3. *The function $u(t, x)$ is the smallest viscosity supersolution of (11).*

Proof. Consider another viscosity supersolution of (11) denoted by $\bar{u}(t, x)$. By the definition, we have for each $(t, x) \in (0, T) \times \mathbf{R}^n$, for any $(p, q, X) \in \mathcal{P}^{2,-}\bar{u}_*(t, x)$, then

$$\Phi(x, \bar{u}_*, \sigma^T(x)q) \geq 0 \text{ and } -p - F_0(t, x, \bar{u}_*, q, X) \geq 0.$$

So for $\alpha = 1, 2, \dots$

$$-p - F_0(t, x, \bar{u}_*(t, x), q, X) - \alpha \Phi^-(t, x, \bar{u}_*, \sigma^T(t, x)q) \geq 0,$$

which follows that $\bar{u}_*(t, x)$ is also a viscosity supersolution of (10). While $u_\alpha(t, x)$ is a viscosity solution of (10), then

$$u_\alpha(t, x) \leq \bar{u}_*(t, x) \leq \bar{u}(t, x).$$

By the limit property in (9), we have $\bar{u}(t, x) \geq u(t, x)$. And the result follows.

For the uniqueness of viscosity solution, we have following result.

Theorem 3.2. *Under assumptions (2) and (4), we assume that for each $R > 0$, there exists a function $m_R : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, such that $m_R(0) = 0$ and*

$$|g(t, x_1, y, p) - g(t, x_2, y, p)| \leq m_R(|x_1 - x_2| (1 + |p|)),$$

for all $t \in [0, T]$, $|x_1|, |x_2| \leq R$, $|y| \leq R$, $p \in \mathbf{R}^d$. And Φ is strictly increasing in y for each (t, x, z) . Then the constraint PDE 11 has at most one locally bounded viscosity solution.

Proof. The proof is done by the same techniques in theorem 8.6 in [3], so we omit it. \square

Remark 3.1. *The constraint satisfies assumptions in this theorem, if $\Phi(t, x, y, z) = y - h(t, x)$, here $h(t, x)$ may be a discontinuous function with certain integral condition. In fact such constraint introduces a reflected BSDE with a discontinuous barrier $h(s, X_s^{t,x})$, c.f. [7]. Another example is solution y reflected on function of z , i.e. $\Phi(t, x, y, z) = y - \varphi(t, x)$, where φ is a Lipschitz function on z .*

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